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A Homological Foundation for Scale Problems in Physics

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Abstract

First steps towards a discrete theory of observation are developed by using algebraic topological concepts, and are shown to account for finite limits of resolution. The cohomology ring on a basic simplicial complex is claimed as the natural language of physical theory. This is illustrated by specific examples and a complete set of generators for such a ring will be called the *Eddingtonian* of a system.

The concept of tensorial covariance is logically replaced by that of an isomorphism between base homologies induced by a bijective simplicial map. This is applied to velocity space to produce an abstract form of the Lorentz-Einstein relativity theory and also to discuss an experimental diffraction arrangement devised by Bohm and Aharonov. The latter is shown to be homologically equivalent to the problem of the red shift, and consequently throws light on the question of galactic recession.

1. *Introduction*

Two discrete algebraic theories of the limits on observation in quantum theory and cosmology which have been put forward respectively by the writers (Atkin, 1968; Bastin, 1966) will be shown in this paper to have a common mathematical basis.

By a *discrete theory* we mean one in which the items of knowledge or information about a physical situation or physical object in given circumstances constitute a finite set. Current physics is a *continuum theory*, and what we mean by this can be expressed in the foregoing terminology, as follows. A *continuum theory* is a theory in which it is assumed that by refining the experimental arrangement (i.e. the circumstances of measurement or of observation) we can obtain a set of higher cardinality, and that this process of refinement can be extended indefinitely. Any discrete theory has to contend with the deep-seated classical assumption of continuity; this assumption has the effect that difficulties in the construction of point-spaces are usually not taken seriously—it is assumed that they can safely be ignored. In this paper we question this assumption and ascribe the existence of limits of observation in the quantum domain and in cosmological systems to its invalidity.

Given a finite set we require, on the one hand, that new sets of greater cardinality (though not necessarily indefinitely greater) can be derived from it and, on the other hand, that some suitable structure associated with the set possesses observational significance, and that this structure persists in the new set independently of its cardinality. We build on the structure we have already established about an object. For example, in the classical idea of the curvilinear motion of a point-particle an important structure is that of a simple ordering, and this must persist when we increase the cardinality of the set of points which constitute the line.

The structures derived from physical observation will be represented in this paper by simplicial complexes and their associated homological and cohomological algebras. This representation will give us a natural way of describing progressive resolution, or refinement of observation, and the limits thereto.

Two physical problems which would normally require continuum dynamics and which will have a paradigmatic role in future developments of the present theory will be solved.

2. *Discrimination*

Bastin and Parker-Rhodes formulated a discrete theory which was capable of representing refinement of observation by exploiting the idea of *discrimination*. This idea was considered to be the expression in logical terms of the universal operation which results in the generation of physical entities and which proceeds according to definite rules which then necessarily cover the process we know as observation. If there exists a set of entities already generated, then two entities in the set may be related under the operation of discrimination, as a result of which a record exists that those entities are unlike. If they are not unlike then no discrimination process takes place. The result of the operation on those two entities is itself a member of the set which may or may not already exist, and upon which discrimination may take place. Two general consequences follow:

(1) The sets of discriminable entities are generated in a hierarchy of stages.
(2) The epistomological innovation of quantum theory, according to which the obtaining of information by observation itself affects the physical system, has a counterpart in the method of construction of point sets by the discrimination process.

The definition of discrimination as the operation of symmetric difference on ordered sets of binary entities led to a physical interpretation for those subsets which were closed under discrimination. A *hierarchy* of discriminable sets was therefore established.

The idea of discrimination as a basic physical process is not new in physics. Rosenfeld (1949) has pointed out that the spin vector has a natural interpretation as a means of establishing dichotomous choice. He calls it a *dichotomic variable*. Rosenfeld would have to find a way of regarding all other quantum theoretical variables on the same pattern if this observation were to be more than a curiosity, and he makes no such attempt. This, however, is just what we want to do, and Rosenfeld's observation is significant for us because it means that we can link our discrimination idea with existing physics at an early stage.

The approach to discrete theory followed by Atkin (1965) began with a development of the intuitive notion of a scale as a generalisation of such simple scaling devices as a mercury thermometer—and the peculiar significance of the bilinear relation in the mathematics associated with it. The pattern of such a scale was then introduced into mathematical physics (Atkin, 1968) via the concept of a Cech homology on compact manifolds.

Our aim now is to explain how the discrimination process may be seen to determine the choice of p-cycles in a certain simplicial complex and how to use this complex, with its homological structure (Hilton & Wylie, 1962), to define an abstract 'measurement situation', or *scale*.

We attach to the set of observations (and here we shall refer to the members of the set as *points*) one more 'point', namely, the complement of the set. It seems appropriate to refer to this as the *antipoint*, and we shall denote it by $\varnothing$. Since the set of entities is defined by the physical process of discrimination there is no mixture of types involved; the 'set' $\varnothing$ contains non-discriminable members and therefore has cardinality unity. Let us now suppose that the cardinality of the original set of observations, without the antipoint, is n. Then, denoting this set by S, we can see that the process of discrimination is a physical means of defining a totally *disconnected simplicial complex* $K_0(S)$. The number of topological components is n, and this is also the *zero-order Betti number* β_0 $(= n)$. This in turn is equivalent to saying that n independent 0-cycles $[z_0{}^i;\, i = 1,\ldots,n]$ are discriminated (or 'seen') in the given measurement situation. These $z_0{}^i$ are labels for the points of S; we shall denote the whole set of them by Z_0 when we wish to emphasise the view of the points as 0-cycles.

Now we ask whether subsets of S—pairs, triples, etc.,—can be discriminated. Such observational possibilities require new levels in the discrimination process, an extension of the discriminatory ability. First let us look at the situation when a particular *pair* of points, A and B, have been discriminated—in the sense that we can observe the set $[A, B]$ containing two unordered members, so that in the first instance A and B are known to be members of Z_0. We then identify the discrimination of A with that of the equivalent pairs $[\varnothing, A]$ and $[A, \varnothing]$ in the extended set $S \cup \varnothing$. It follows that discriminating the pair $[A, B]$ is equivalent to determining a typical 1-cycle in a simplicial complex $K_1{}^+(S)$ via the formal combination

$$z_1 = [\varnothing, A] + [A, B] + [B, \varnothing]$$

The simplicial complex $K_1{}^+$ is obtained from K_0 by

(i) augmenting the vertex set Z_0 by forming the union $Z_0 \cup \varnothing$
(ii) adjoining to the new vertex set the 1-simplices defined by the pairs $[\varnothing, A]$, $[A, B]$ and $[B, \varnothing]$.

In general we would expect this extended discrimination process to determine a whole set of 1-cycles, Z_1, on a suitable $K_1{}^+$; the cardinality of Z_1 being β_1, the first-order Betti number of $K_1{}^+$. (This is making the assumption, which we shall adhere to in this paper, that the homology groups are torsion free.) Furthermore we would expect a whole series of p-cycles, for various integral values of p, giving us the sets $Z_0, Z_1, \ldots, Z_p, \ldots, Z_{n-1}$—each p-cycle corresponding to the discriminatory ability to determine a subset of $(p+1)$ points of Z_0.

We now adopt the discriminable p-cycle as the mathematical concept in our theory which corresponds to the physical observation. We shall use the word *object* (or p-object) to describe any observable p-cycle. We have, of course, a problem of reconciling this mathematical treatment of observation by discrimination with the elementary logical operation by which discrimination was introduced. This problem is the counterpart of the notoriously intractable measurement problem in current quantum theory. It is possible that the difficulty in our case is easier to come to terms with however, just because we encounter it at such a rudimentary stage in our theory. In both cases we appear to be confronted with the difficulty that the elementary interactions in terms of which the theory is formulated are microscopic, whereas measurement is macroscopic and has to be constructed from a multiplicity of microscopic interactions by some means, statistical or other. In current theory there seems little one can do about the micro-macro conflict, but in the approach we are now proposing physical magnitude is still undefined. Logically elementary operations and physically elementary ones cannot be differentiated by magnitude therefore, which requires the whole hierarchical structure for its understanding. The difficulty in our case—accordingly—is one for the imagination, rather than one which threatens logical coherence.

When $p = 0$ the object is a point. We notice also that when we see a p-cycle z_p we do not see any z_q, with $q > p$. More precisely, we discriminate the set of p-cycles in a complex $K_p{}^+$ and at this stage we do not permit the observation of any $K_q{}^+$ for $q > p$. The point of this, in the mathematical theory, is that, otherwise, we could not be sure that the p-cycle is not a p-boundary, and that Z_p is not empty (Dowker, 1952).

The total number of objects discriminated is

$$N = \beta_0 + \beta_1 + \cdots + \beta_{n-1}$$

and this does not include the antipoint $\varnothing$. By using the existence of this upper bound we can identify a state of *maximum discrimination* as one in which all the possible subsets of Z_0 can be discriminated. When this occurs $\beta_p = {}^nC_{p+1}$ and

$$N = {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n - 1$$

These developments lead us to identify a class of related physical objects with a set of *homology groups* $H_p(K^+)$ on a suitable complex (Hilton & Wylie, 1962), and the objects discriminated by S are generating elements of these homology groups. The complex K^+ [or $K^+(S)$ when we wish to specify the given experimental arrangement] possesses $\beta_0 + 1$ verticles and is embedded in an abstract polyhedron possessing these same vertices.

Our discussion has been based on the simplifying assumption that orientation of the complex is ignored, and in this case the appropriate chain complex is described by using J_2 (the integers mod 2) as coefficients. Subsequently we shall suppose that the coefficients are in J (the integers). This will allow not only for the possibility of K^+ possessing orientation but also for the very natural role which we shall assign to the associated cohomology groups. The Universal Coefficient Theorem (Hilton & Wylie, 1962) ensures that there is no loss in the correspondence we are setting up between structure and observation.

3. *Resolution of Scales and Hierarchical Structure*

Hierarchical structure arises out of an attempt (at the observational level) to increase the cardinality of Z_0 through attempting to answer the following question. Given a scale S, with card $Z_0 = n$, is it possible to find a scale S' which *resolves the objects* of S in the sense that these objects are to be seen as the points of S'? If S' exists we shall refer to it as a *resolution* of S, and we see at once that if card $Z_0' = n'$ then in the special case of maximum discrimination for S, $n' = 2^n - 1$ (Atkin, 1971; Bastin, 1966).

The important result shown by Parker-Rhodes, namely, that the process of resolution must cease after a finite number of steps, can now be seen as a limit to the extent to which we may permit S' to be S itself. If S possesses only a *minimum ability to record* its various p-cycles (a minimum memory-store capacity) then we have to ask how often can it redefine its objects so that they become points.

At the first level of operation, suppose that S is seeing the p-cycles Z_p. Then it is effectively picking out sets of $(p+1)$ points from Z_0. It may clearly do this, and keep a record of them, if it can order the original n points, and so attach ordinal numbers to the members of the $(p+1)$ set. This would only require a minimum of n places in the memory; these, together with the ordering attached to them, would suffice to record all the members of the Z_p. Since p takes n values we deduce that the minimum memory space for the whole operation on the basis that the original memory space were fixed, would be n^2. We therefore have the result: *if S possesses maximum discrimination and minimum recording ability, resolution is possible if $n^2 \geqslant 2^n - 1$.* This will be called *maximal resolution.* We can proceed further in this way to construct further levels of discrimination on the basis that reference is always made back to the original memory store, and this basis provides a particularly realistic type of hierarchy.

Commencing with the case $n = 2$ we obtain the sequence

$$2 \quad 4 \quad 16 \quad 256 \quad \text{stop}$$

for the memory and

$$3 \quad 7 \quad 127 \quad \sim 10^{38}$$

for maximal resolution, the latter giving us the cumulative sums

$$3 \quad 10 \quad 137 \quad \sim 10^{38} \quad \text{stop}$$

If we remove the condition of maximum discrimination but retain the minimum memory store we obtain the sequences

$$n \quad n^2 \quad n^4 \quad \cdots$$

and

$$B \quad B' \quad \cdots$$

where $B = N$, and the B', etc. are suitably defined.

The sequences of cardinalities concerned in maximal resolution are likely to emerge in observational situations because of the orientation of physics towards refinement of description wherever this is possible. Working from the conventional view of physical magnitudes various writers have stressed the central importance of the coupling constants of the different physical fields—because of the way in which these dimensionless numbers fix the values of the natural atomic and cosmological constants, and hence of the natural units. This view is particularly associated with Eddington (see, for example, *Relativity Theory of Protons and Electrons*, 1936, for an early statement). The calculation of the sequence for $n = 2$ above is not, of course, that put forward by Eddington.

4. *The Ring of Cocycles and Orthodox Measurement*

The central feature of orthodox methodology in physics is that the experimentalist refers all his observations to a common backcloth of observation. We shall represent this backcloth by a *base homology* on a background simplicial complex (which we denote by BK^+). This base complex is used to carry various physically intuitive notions. For example, the vertices of BK^+ (the points of the scale S) may be associated with the physically intuitive notion of 'geometrical point', or again, each may be associated with the physically intuitive notion of 'particle'. The 1-cycles of BK^+ (defined by pairs of points on S) may be associated with the physically intuitive notion of velocity, linear momentum, plane wave motion (where we shall see that the wavelength idea is an attempt to characterise the point-pair separation), or geometrical line. The 2-cycles of BK^+ may be associated with the notion of geometrical plane, angular momentum (the triangle with orientation), vector torque, electric or magnetic flux. The 3-cycles of BK^+

may be associated with the notion of geometrical volume, a three-dimensional body in mechanics, or electric charge density.

We have now, however, to make an important distinction. The base complex defines a kind of abstract structure for these physical concepts.† Thus it may describe magnetic field to the extent of specifying the structural complexity of that concept. We have said something vital about magnetic field when we have specified it in terms of the motion of a test-particle and required that the acceleration of the test-particle due to that field shall be at right-angles both to the direction of the field and the motion of the particle. It is not usual, however, for us to abstract this knowledge from such knowledge as the extent or degree of uniformity of the field, and indeed all the detailed knowledge that the physicist is accustomed to lump with it. Because this distinction is unfamiliar it is difficult. (It is not even certain that the concept of a test-particle will be interpreted the right way and will not evoke irrelevant questions concerning the structure of the test-particle or its practical feasibility.) Having established the distinction we now have to provide the detailed application of the abstract structure.

Each of the physical notions associated with a particular observable p-cycle is to be represented in a precise mathematical way via a *mapping* which homomorphically maps that p-cycle into a suitable coefficient group. This means that the ordinary language of theoretical physics is to be the *cohomology* (Hilton & Wylie, 1962) of the base complex BK^+, and we select the coefficient group, in the first instance, as the additive group of integers J. Thus every physical observation is to be represented in the theory by the *cocycle* mapping

$$z^p : Z_p \to J$$

on the relevant group of p-cycles, Z_p. These possible p-cocycles Z^p act as generators for the cohomology group $H^p(K^+;J)$, and we know that in the finite cases under consideration these groups are isomorphic to the homology groups $H_p(K^+)$. It is well known also that, since J possesses a ring structure, this graded cohomology group $H^{\cdot}(K^+)$ also possesses a ring structure in which multiplication obeys the commutative law

$$z^p . z^q = (-1)^{pq} z^q . z^p$$

where $z^p . z^q$ is a cocycle in $H^{p+q}(K^+;J)$.

As a matter of notation, the *value* of a cocycle z^p on a particular cycle z_p is usually written as (z_p, z^p); this is an integer in J. Thus, although all values of physically observable things are 'merely' numbers, their essential differences are contained in the cocycles which map into these numbers. There is a qualitative difference between the physically intuitive notions which is expressed in the orders of the cocycles which represent them. We see too that our theory contains in it the possibility of emphasising the 'value of a measurement' (an essentially classical view) or of emphasising

†Bastin & Kilmister (1952) introduced the expression 'theory-language' to describe the abstract structure of the interrelated concepts of mechanics and electromagnetism.

the observable as an *homomorphism* (or linear map), the latter view being essentially that of quantum theory.

In the representation of a particular physics (Atkin, 1965), such as the dynamics of a set of classical bodies, the total information of the system will be contained in a *complete set of cocycle generators* which are defined on BK^+. This set of generators will therefore define a ring of cocycles, and this ring will contain all the essential algebraic structure of the dynamics. Indeed, 'dynamics' for this system will be defined by this algebraic ring. We propose that such a complete set of cocycle generators for a particular system be called the *Eddingtonian* of the system.

But what has happened, we might ask, to the ordinary experimentalist's idea of measuring a continuous variable—such as a galvanometer reading—in the laboratory. In the first place we must point out that the experimentalist is never able to observe a continuous variable, that is, continuous in the mathematician's sense. When the physicist represents his galvanometer reading by a *real number* he imagines the following procedure to be justifiable, namely,

(i) BK^+ possesses an *uncountable* set of vertices Z_0; precisely, that card Z_0 = the cardinality of the continuum of the reals, **R**.
(ii) observation of the galvanometer current amounts to a map

$$I^0 : Z_0 \to \mathbf{R}$$

In fact this procedure is operationally unrealistic. The actual experimental vertices of BK^+ are finite in number and are always determined by some choice implicit in the experimental method which effectively defines a limiting unit of observation. One may think of this limiting unit as the practical limit of resolution which selects a just-distinguishable pair of points. This is not to attribute absolute significance to the limit, only to stress that an experimental arrangement carries something like a characteristic length which determines the scope of its applicability. It must not, of course, be confused with the unit of 1 cm or 1 amp which is marked on the instrument.

This limiting unit actually defines a 1-cycle z_1 in the base complex BK^+ and is mapped into the coefficient group J by the 1-cocycle (say) I^1 which represents the object called 'current'. There will be a cocycle generator (say) $\hat{I}^1$ such that every observation corresponds to the cocycle $I^1 = n\hat{I}^1$, for a specific integer n. This $\hat{I}^1$ is the Eddingtonian for this particular observation. Incidentally, we notice that if we select some *real number* h to denote the value of the cocycle generator then the cocycles I^1 are being mapped into a coefficient group which is algebraically isomorphic to J. In our theory the coefficient group is to be J *up to isomorphism*.

Finally, we notice that the orthodox view of measurement contains another consequence which can be expressed in homological terms. This is the consequence of supposing that resolution is always possible and that it can be repeated an indefinite number of times. This means that all subsets are supposed to simultaneously discriminable (the use of the reals **R** is already a claim to this possibility, since they are in one-to-one correspondence with all subsets of the integers). Consequently the process of discrimination, as we have described it, does not contain the condition 'when S sees the p-cycles Z_p it does not see any z_q for which $q > p$'. This means that, in the orthodox view, every cycle is a boundary in the homology theory, and the base homology is therefore trivial; $H_p = 0, p > 0$.

It has been shown (Atkin, 1968), and will be discussed in more detail in a later paper, that natural laws in classical theoretical physics are illustrations of this assumption of trivial base homology when the base complex is supposed to represent space-time.

5. *Interpretation of Cocycle Generators on the Base Homology*

The present section examines the base homology of ordinary physics by a consideration of the Eddingtonian of several specific experimental situations. We look for the cocycle generator which is inherent in orthodox measurement situations, reminding ourselves that the values of a z^p on a z_p are in a ring isomorphic to J.

Case 1. *Length*

Usually the measurement of length l cm is presented as $(l \pm \Delta l)$ cm, where Δl denotes the so-called 'error'. In fact, Δl is a number attached to the *least resolvable point-pair*, under the given experimental conditions. More precisely, this least resolvable point-pair must be the basic 1-cycle $\hat{z}_1$ in the base complex BK^+, and the Δl is the value of the notion of length (in **R**) associated with that 1-cycle. Thus Δl is associated with the required 1-cocycle generator $\hat{z}^1$ of the length-measuring process.

A significant dimensionless quantity associated with the length measurement is the so-called *relative error* $\Delta l . l^{-1}$; accordingly, we define the cocycle generator $\hat{z}^1$ as the map whose value on $\hat{z}_1$ is the real number

$$(\hat{z}_1, \hat{z}^1) = \Delta l . l^{-1}$$

Then any length measurement, L, will be a cocycle z^1, where $z^1 = n\hat{z}^1$ for a suitable $n \in J$. Of course, the physical role of the Δl is to ensure that $l = m(\Delta l)$, for a suitable $m \in J$ (which amounts to saying that measuring length is equivalent to counting packets—or quanta—of Δl). Then measuring L amounts to evaluating (z_1, z^1), where z_1 is the 1-cycle (the point-pair) defined on BK^+ by the end-points of the 'thing' that has L ascribed to it. This z_1 will be homologous to the basic cycle $\hat{z}_1$, and so we get

$$(z_1, z^1) = n(\hat{z}_1, \hat{z}^1) = n(\Delta l . l^{-1}) = \frac{n}{m}$$

Thus, measuring length on a geometrical axis amounts to mapping point-pairs into pairs of integers (n, m). These pairs of integers are symbols in the field of quotients of J, that is to say, in the rationals Q. Also, if $k \neq 0$ and $k \in Q$, we have that the length (n,m) satisfies the projective relation

$$k(n,m) = (kn, km) = (n,m)$$

Furthermore, any length L can be represented symbolically over the two-dimensional module spanned by $(0,1)$ and $(1,0)$, via

$$L \to n(1,0) + m(0,1)$$

and the physical measurements which correspond to the base vectors $(0,1)$ and $(1,0)$ are

$$(0,1) \to \text{length observed is } \Delta l \text{, the least resolvable}$$

and

$$(1,0) \to \Delta l = 0$$

If we accept the latter condition as applicable to our geometrical axis we are actually saying that we believe that an infinite (although countable) sequence of subdivisions of a line is physically realisable. With this assumption it would follow that there is a bijective relation (a one-to-one onto mapping) between all measurable lengths and the rationals, and this implies that the *geometrical axis* appears as a *projective line* P^1. It therefore possesses the homology

$$H(P^1) = H_0(P^1) \oplus H_1(P^1)$$

with $H_0 = J$ and $H_1 = J$. In particular it possesses a single basic 1-cycle ($\beta_1 = 1$) which generates J.

Case 2. *Wavelength Attributed to a Light Signal*

From the analysis of length measurement we see that the wavelength λ cm plays the role of Δl—the least resolvable 'distance' under the given experimental conditions. The cocycle generator $\hat{z}^1$ associated with this object will then be the dimensionless number λ/λ_0, where λ_0 will be a characteristic length. In what follows we shall make the simplifying assumption that $\lambda_0 = 1$ (which can be seen as defining the unit 1).

Case 3. *Diffraction Grating*

The observed plane is spanned by two geometrical axes whose independence means that we may form their cartesian product to obtain the base complex. One axis is characterised by the light signal with a cocycle generator $\hat{z}^1$ (whose value on the basic $\hat{z}_1$ is taken as λ), and the other by the grating length (say, L) and cocycle generator whose value is $\Delta L . L^{-1}$

The plane is therefore $P_L{}^1 \times P_\lambda{}^1$, where (for example) $P_L{}^1$ denotes the projective line specified by the grating length. Now, by the Kunneth Formula (Hilton & Wylie, 1962) we deduce the homology of the base complex as

$$H(P_L{}^1 \times P_\lambda{}^1) = H_0 \oplus H_1 \oplus H_2$$

where $H_0 = J$, $H_1 = J \oplus J$, and $H_2 = J$.

Since the homology is isomorphic to the cohomology we deduce that there is a 2-cocycle generator which takes a value equal to the product of the two separate 1-cocycle generators (because of the definition of the cohomology ring). The two 1-cocycle generators take values $\Delta L.L^{-1}$ and λ respectively, so that we get the value of the 2-cocycle generator as

$$(\hat{z}_2, \hat{z}^2) = (\Delta L.L^{-1}).(\lambda)$$

But the plane is also defined as the cartesian product of the D-axis (distance of the screen from the grating) and the fringe pattern x-axis (x denoting the typical fringe width). In this case the two 1-cocycle generators take values $\Delta D.D^{-1}$ and x. We therefore deduce, since $\hat{z}^2$ is unique,

$$\Delta L.L^{-1}.\lambda = \Delta D.D^{-1}.x$$

and provided L and D are measured under circumstances which permit us to take $\Delta L = \Delta D$, we obtain

$$\frac{x}{D} = \frac{\lambda}{L}$$

The apparent simplicity (even obviousness) of this calculation may mislead. In the present paper all the physical situations (with the partial exception of the galactic recession) involve only topologies which fit our ordinary views of space and time. We have in principle, however, opened up the possibility of changing the base complex BK^+, and when this is exploited we shall be confronted with cases which give counter-intuitive results. In the analysis of this section, the fact that the concept of wavelength has been introduced does not mean that the whole propagation theory has been assumed. On the contrary, the homology theory intersects with the familiar wave propagation theory only to a limited extent.

6. *Simplicial Maps between Base Complexes K and K′*

If we imagine a change in the experimental conditions of observation then we would need to study the relation between the two base complexes involved. In particular the points which are observed (the 0-cycle objects) constitute the sets K_0 and K_0', and we have seen how the observation process builds on these sets to discriminate higher-order p-cycle objects. However, since it is the homology of the base complex which is physically significant it follows that we are particularly interested in the effect on the homologies as we change from one base complex K to another K'.

The mathematical transformations which relate K_0 to K_0', and subsequently K to K' and $H(K^+)$ to $H(K'^+)$, are the logical precursors of those transformations of space-time which ensure Lorentz or Riemann invariance. The situation which corresponds to, say, Lorentz invariance must be one in which the transformation (or mapping) which relates K_0 to K_0' induces an *isomorphism between the homologies* $H(K^+)$ and $H(K'^+)$. These mappings will be special cases of what are called *simplicial maps* (Hilton & Wylie, 1962).

Precisely, a simplicial map ψ is a map $\psi: K_0 \to K_0'$ (that is, it maps the vertices of the complex K into the set of vertices of K') which satisfies the condition: 'whenever the vertices $[a_0, a_1, \ldots, a_p]$ in K_0 define a p-simplex in K, then the vertices $[\psi a_0, \ldots, \psi a_p]$ in K_0' define a q-simplex in K', for some $q \leqslant p$'. It can be shown (Hilton & Wylie, 1962) that such a map induces a homomorphism (which is of course a linear transformation) between $H(K^+)$ and $H(K'^+)$; this happens because ψ induces a well-behaved correspondence between the cycles of K and the cycles of K'. This induced homomorphism is commonly written as $\psi_*: H(K^+) \to H(K'^+)$, and consists of a series of maps $\{\psi_{*,p}\}$ such that each $\psi_{*,p}$ is a homomorphism between $H_p(K^+)$ and $H_p(K'^+)$, $p > 0$.

The traditional idea of invariance corresponds, in this context, to the idea of an *isomorphism* $H_p(K^+) \simeq H_p(K'^+)$. Physically this means that there are just as many p-cycle objects in K as in K': the changes in the experimental conditions have not created nor annihilated any observational objects, nor have they changed a p-cycle into a q-cycle. It does not mean, however, that there is necessarily an identity between the generators of $H(K^+)$ and $H(K'^+)$. A common example in orthodox theory is to be found in a rotation of the geometrical axes of euclidean space.

Now this isomorphism can be shown to occur when the simplicial map $\psi: K_0 \to K_0'$ is *bijective*, that is to say, ψ is (1–1) and onto. Precisely, ψ satisfies the conditions

(i) $\psi a_r = \psi a_s$ implies $a_r = a_s$, for any $a_r, a_s \in K_0$, and
(ii) for any vertex $a' \in K_0'$ there exists some vertex $a \in K_0$ such that $\psi a = a'$.

Thus, a bijective simplicial map ψ induces an isomorphism ψ_* between the base homologies. Since, moreover, a large part of physical observation consists in observing essentially the same base complex, bijective simplicial maps between K_0, K_0', K_0'', ..., etc. can be expected to be of importance. Since the cohomologies and homologies are isomorphic, it follows that a bijective simplicial map will also induce an isomorphism between the cohomologies on the two base complexes. It is these bijective maps and the corresponding linear transformations between the higher-order sets of p-cocycles which correspond to the more usual theories of tensors and their covariance.

Let us take as an example two complexes K and K' which are observed geometrical axes possessing the homological structure of the projective line

P^1. Then the coordinates x, x' (where these are rational numbers) denote the points of K_0 and K_0', respectively. Since Q is a division algebra we can use the mathematical result that the most general (1–1) correspondence between these lines, which is also a bijection, is the *bilinear relation* (also known as an *homography*)

$$Axx' + Bx + Cx' + D = 0$$

where $A, B, C, D \in Q$ and where $AC - BD \neq 0$.†

We notice that if the above bilinear relation is degenerate (i.e. when $AC - BD = 0$) then the relation between x and x' is not (1–1) onto, and so the base homologies cannot be isomorphic. Also, any other relation which is not (1–1) must correspond to a lack of isomorphism in the base homologies; for example, a relation of the form

$$Ax^3 x' + Bx^2 + Cx' + D = 0$$

means that there are generally three points in K_0 mapped onto one point in K_0'. We see too that the homography above leads in a natural way to the consideration of a *quadratic form*—when it is meaningful to allow $x = x'$. The linear maps from K_p to K_p which preserve this quadratic form will now be compatible with the isomorphisms induced by the bilinear relation.

An important example of the bilinear relation occurs in the velocity space of Lorentz–Einstein relativity theory. The unique status of the velocity of light becomes an expression of the bijective nature of the simplicial mappings which are allowed. The two sets of velocity measures v, v' along parallel axes which move with a relative velocity V amount to two base complexes K_0, K_0' being observed by two observers S and S'. We therefore consider the bilinear relation

$$Avv' + Bv + Cv' + D = 0$$

The final form of this relation is obtained by using the observations

(i) $v = V, v' = 0$
(ii) $v = 0, v' = -V$
(iii) $v = v' = c$

whence

$$Vvv' + c^2(v - v') - c^2 V = 0$$

We notice that the double points of the homography occur at $\pm c$; only one numerical value is therefore possible for this double point. This unique status for c is not merely an accident of observation.

If, in addition, the observers S and S' represent v and v' by the rational ratios x/t and x'/t' then the homography is equivalent to the separate relations

$$x' = \beta(x - Vt) \qquad \text{and} \qquad t' = \beta\left(t - \frac{xV}{c^2}\right)$$

†The role of the homography in classical physics was analysed by Atkin (1965).

where β is a separation constant, independent of x, x', t, t'. This is actually the first appearance of length and time variables x, t or x', t'. Unlike the velocities, they are undefined in this problem, and have no experimental meaning until they are given it, as we are now doing. The x and t occur in pairs for each v, and define 1-cycles to replace the 0-cycles. It is an important characteristic of our approach that we are allowed to identify cycles with velocity relationships in, for example, the problem at present under discussion and yet to identify cycles with lengths in the diffraction grating (Case 3). In neither case do we imagine that we are automatically setting up a manifold of physical operations which automatically includes the other. In so far as we need such a concept we have to set it up explicitly and exactly as we need it.

x, t and x', t' are contained in the linear map L defined by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = L\begin{pmatrix} x \\ ct \end{pmatrix} \stackrel{\text{def}}{=} \beta\begin{pmatrix} 1 & -V/c \\ -V/c & 1 \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix}$$

and if we appeal to the fact that the roles of S and S' are reversed when V changes its sign, we obtain the condition

$$L(-V) = L^{-1}(+V)$$

and this results in the condition

$$\beta^2\left(1 - \frac{V^2}{c^2}\right)^{-1} = 1$$

We see too that the bilinear relation naturally leads to the condition on the quadratic form, namely,

$$x^2 - c^2 t^2 = 0$$

Allowed transformations, in space-time, must henceforward preserve this relation—a situation which is more general than saying that the *quadratic form* $x^2 - c^2 t^2$ must be preserved. The invariance of the geometrical *conic* rather than of the *form* is in fact the basis of Weyl's (1918) gauge theory—which therefore rests on the idea of bijective simplicial maps.

We now consider certain cases which illustrate the isomorphism between $H^1(K^+)$ and $H^1(K'^+)$ consequent upon a change of base complex. Each isomorphism is expressed in terms of easy linear maps.

Case 4. *Modified Diffraction Grating*

In an experiment which has been discussed by Aharonov and Bohm, the beam of light, in an ordinary diffraction grating arrangement, is replaced by a beam of electrons, and solenoids are placed inside and co-axial with the bars of the grating. The physically significant thing is that there is an interaction between the beam and the magnetic field of the solenoids; this is manifest by a shift in the diffraction pattern. We obtain this change in the pattern by equating the physical dependence of the beam and the magnetic field to a linear dependence of their respective cocycle generators on the base complex. Referring to the notation introduced under Case 3, we now have a base complex which represents the plane system and which may be denoted symbolically by $P_L{}^1 \times (P_\lambda{}^1 + M)$, rather than by $P_L{}^1 \times P_\lambda{}^1$; the symbol M denoting the magnetic field produced by the solenoids. The plus sign is to be interpreted by saying that the cocycle generators take the values

$$\Delta L . L^{-1} \quad \text{corresponding to the term } P_L{}^1$$

and

$$\lambda + \alpha\lambda \quad \text{corresponding to the term } P_\lambda{}^1 + M$$

The cocycle generators for the cohomology groups $H^1(P_\lambda{}^1)$ and $H^1(M)$ are now to be linearly dependent; the term $\alpha\lambda$ corresponds to the latter. The factor α is a rational number, in Q, but it must not be an integer—if the physical dependence between the beam and the field is to be observed.

Applying the argument of Case 3 we obtain the form of the new fringe pattern as

$$\frac{x}{D} = \frac{\lambda + \alpha\lambda}{L}, \qquad \alpha \neq 1$$

We notice that the effect of the magnetic field is solely manifest via its alteration of the cocycle generator at the level $p = 1$. That a corresponding physical interaction in fact exists, demonstrates that $H_1(M) \neq 0$, and this is true independently of the possible values of $H_p(M)$, when $p > 1$. An observed phase shift in the diffraction pattern is an experimental demonstration that the Eddingtonian for the magnetic field contains a generator $\hat{z}^1$ in the ring of cocycles on the base complex. If we then look for a quantity in the orthodox continuum theory of the magnetic field which can illustrate this, we shall need a line integral whose value on a closed curve in the solenoid is not zero. This is the origin of Aharanov's 'modular momentum' and of the choice of the potential integral $\oint \mathbf{A} . \mathbf{dl}$ to play the role of modular variable in this particular problem.

Case 5. *Doppler Effect*

The velocity which we attribute to a moving point, or particle, considered in relation to the light signal which gives us our information about it, specifies a cocycle—a 1-cocycle when the 0-cycles of the base complex refer to geometrical points. If the moving object occupies points A and B, in that order, in a standard unit of time then the light signal 'interacts' with that geometrical 1-cycle in conveying the information to the observer. The interaction manifests itself in the theory, as in Case 4, through a linear map defining an isomorphism between the base complexes. The base complex is finally a combination which is symbolically expressed as $P_\lambda{}^1 + P_0{}^1$, where $P_0{}^1$ denotes the geometrical object-axis, and the 1-cocycle generator for

$H^1(BK^+)$ will now take a value $\lambda + \alpha\lambda$, for a suitable non-integral value of α, in Q.

The Lorentz–Einstein relativity theory (see above) gives, by a well-known argument,

$$1 + \alpha = \sqrt{\left\{\frac{1 + (v/c)}{1 - (v/c)}\right\}}$$

or

$$1 + \alpha = 1 + \frac{v}{c}, \quad \text{when } v \ll c$$

We see that the Doppler effect is homologically the same problem as the modified diffraction grating. Each illustrates the occurrence of a physical dependence at the same homological level ($p = 1$), and this dependence is an expression of an isomorphic change in the base homology.

Case 6. Galactic Recession

This has an immediate interpretation via the Doppler Effect, and this is the basis for the orthodox view that the red shift $(\lambda' - \lambda)/\lambda$ must be interpreted by saying that the source has a velocity v $(= L/T$, say) away from the grating. This is usually expressed as, for example,

$$\frac{\lambda' - \lambda}{\lambda} = \frac{v}{c} = \frac{L}{cT}, \quad \text{when } v \ll c$$

T being the Hubble constant.

We can now see that this shift in wavelength is due to an isomorphism between $H^1(P_\lambda{}^1)$ and $H^1(P_\lambda{}^1 + P_0{}^1)$. It is not necessary for this 1-cocycle generator in $H^1(P_0{}^1)$ to be due to velocity, although, of course, it is certainly sufficient to say so. (The experiments of Pound & Rebka (1960) on the Mossbauer effect have similarly illustrated the above process with the interaction of gamma rays with a gravitational field—at the 1-cocycle level.)

The case of the galactic recession is of very great importance, because the physical circumstances that would make the conventional successive approximation methods plausible do not exist (see McCrea's discussion of whether or not the galactic recession indicates a systematic information cut-off analogous to that indicated by the existence of Planck's constant). Hence the galactic case is one which fits the homology theory naturally and the conventional theory not at all. More fundamentally this case provides quantisation in the sense that it introduces a numerical value for the galactic constants in terms of the microscopic ones, through the dimensionless constants in the manner which Eddington conjectured should be used, but employing a system of dynamics within which such an appearance of a pure number with experimental interpretation is credible.

The number which arises as the order of the largest discriminable set in the hierarchy of resolutions ($N \sim 10^{39}$) is a limit to the possibility of extending the base homology. It arises when the base homology is used to define a method of measurement and when progressive refinement of experiment is pushed to the limit. In the current discussion of galactic recession this number is automatically interpreted as a length ratio (one may speak of the ratio of the electron radius to the radius of the universe). By applying the interaction theory we have developed for the Bohm/Aharonov experiment we get the following picture.

(1) Conventional theory ascribes a recession as the interpretation of the red shift.

(2) Correct analysis of the homological structure shows the existence of a second cocycle generator, and therefore one could have an equally physically correct account with a cosmical electrodynamical effect, exactly by analogy with the Bohm–Aharonov case.

(3) The strength of the interaction is to be specified in the cohomology on the base complex—which remains isomorphic to itself. There is actually a lower bound to the interaction strength which corresponds to the true picture being a linear combination of two cocycles. Hence it is not possible for the physicist to assert that the recession picture is physically the real one, because what he means when he says this is that he can take a new embedding complex to get rid of the physical interaction. It is just a part of our convention that we always do this when we are dealing with space (or length) but that we preserve the interaction when we have an electromagnetic case to deal with. It does mean, however, that we are at a loss to understand the cosmical information limit.

Our treatment recalls an original argument that was advanced in favour of the steady-state theory. It was argued that if the universe were not in a steady state then we ought to contemplate such a wide range of possible laws that we should have no fixed starting point. This contention was plausible but one could not further deduce—as one would wish—from the implausibility of such chaos, a case for the steady state. The argument that the steady-state theorists were sensing, but lacking, was the possibility of arguing that the steady-state model specified a more correct base homology than the current (implicit) homology. For it is impossible to present the galactic recession as an interaction between 1-cocycles if the base complex is a sphere (which has no 1-cycles).

It is also possible to regard the cosmological tradition which started with Milne as having been concerned with the choice of a base complex. A recent exponent of this tradition, Prokhovnik (1970), has used the Milne (1948) substratum hypothesis to review special relativity, and in so doing he makes it especially clear that he gets his results by having the primitive operational statement of special relativity (which is in terms of relative motion with respect to an observer) supplemented by a statement of motion with respect to the substratum. It is very natural to see this change as essentially a search for a non-trivial base homology.

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